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ON THE STABILITY AND OSCILLATIONS OF ANISOTROPIC PLATES

\- USSR -

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ON THE STABILITY AND OSCILLATIONS OF ANISOTROPIC PLATES

/Following is a translation of an article written by S. A. Ambartsumyan and A. A. Khachatryan in Mekhanika i Mashinostroyeniye (Mechanics and Machine Building) No. 1, 1960, Yerevan, pages 113-122./

1. Let us investigate a rectangular plate of uniform thickness h, with one of the elastic symmetry planes parallel to the middle plane of the plate, the other two being parallel to its sides.

The rectangular coordinate system (α, β, γ) is so selected that the coordinate plane $\alpha\beta$ coincides with the median plane of the plate, but the coordinate axes α, β, are in direction of its sides.

The following hypotheses are assumed /1/ :

a) The normal stresses σ_γ on areas parallel to the median plane may be neglected as compared to the other stresses.

b) The normal distance (γ) between two plate points remains unchanged after the deformation.

c) For tangential stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ we have

$$\tau_{\alpha\gamma}=\frac{1}{2}\left(\frac{h^2}{4}-\gamma^2\right)\varphi(\alpha,\beta), \qquad \tau_{\beta\gamma}=\frac{1}{2}\left(\frac{h^2}{4}-\gamma^2\right)\psi(\alpha,\beta) \tag{1.1}$$

where $\varphi(\alpha,\beta)$ and ψ (α,β) are arbitrary functions to be determined of coordinates α,β.

These hypotheses, as is known /2,3/, prove their validity. Some inconsistency, which is due to the acceptance of hypothesis (a), can be justified in the present case. The fact is that in omitting σ_γ, we first of all commit only small errors for the majority of actual plates, since all terms connected with σ_γ have small multipliers; second -- omission of σ_γ does not distort the qualitative aspect of the investigated problems, and third -- omission of σ_γ substantially simplifies problems of

stability and oscillations of anisotropic plates.

In work /1/, while investigating problems of bending of anisotropic plates, a resolving system of three differential equations in three unknown functions is derived: normal dislocation $w(\alpha,\beta)$ and functions $\varphi(\alpha,\beta)$, $\psi(\alpha,\beta)$. For the accepted notations the system has the form

$$
\begin{gathered}
\frac{\partial\varphi}{\partial\alpha}+\frac{\partial\psi}{\partial\beta}+\frac{12}{h^3}Z=0 \\
\left[B_{11}\frac{\partial^3}{\partial\alpha^3}+(B_{12}+2B_{66})\frac{\partial^3}{\partial\alpha\partial\beta^2}\right]w-\frac{h^2}{10}\left[a_{55}\left(B_{11}\frac{\partial^2}{\partial\alpha^2}+B_{66}\frac{\partial^2}{\partial\beta^2}\right)\varphi+\right. \\
\left.+a_{44}(B_{12}+B_{66})\frac{\partial^2\psi}{\partial\alpha\partial\beta}\right]+\varphi=0 \qquad (1.2) \\
\left[(B_{12}+2B_{66})\frac{\partial^3}{\partial\alpha^2\partial\beta}+B_{22}\frac{\partial^3}{\partial\beta^3}\right]w-\frac{h^2}{10}\left[a_{55}(B_{12}+B_{66})\frac{\partial^2\varphi}{\partial\alpha\partial\beta}+\right. \\
\left.+a_{44}\left(B_{66}\frac{\partial^2}{\partial\alpha^2}+B_{22}\frac{\partial^2}{\partial\beta^2}\right)\psi\right]+\psi=0
\end{gathered}
$$

where Z - intensity of the normal approximated surface load; B_{ik}, a_{44}, a_{55} -- known elasticity coefficients /1,4/.

For setting up equations of static and dynamic stability and ocillations of rectangular anisotropic plates, which are investigated here, we start with the equation system (1.2).

2. We derive the equation of static stability by substituting in (1.2) for Z the expression /4,5/

$$Z=T_1^{\circ}\frac{\partial^2 w}{\partial\alpha^2}+T_2^{\circ}\frac{\partial^2 w}{\partial\beta^2}+2S^{\circ}\frac{\partial^2 w}{\partial\alpha\partial\beta} \qquad (2.1)$$

where T_1^0, T_2^0 and S^0 -- tangential forces per unit length, acting in the median plane of the plate.

Substituting (2.1) into 1.2) we obtain the final equations of stability of an orthotropic plate

$$
\begin{gathered}
\frac{\partial\varphi}{\partial\alpha}+\frac{\partial\psi}{\partial\beta}+\frac{12}{h^3}\left(T_1^{\circ}\frac{\partial^2 w}{\partial\alpha^2}+T_2^{\circ}\frac{\partial^2 w}{\partial\beta^2}+2S^{\circ}\frac{\partial^2 w}{\partial\alpha\partial\beta}\right)=0 \qquad (2.2) \\
\left[B_{11}\frac{\partial^3}{\partial\alpha^3}+(B_{12}+2B_{66})\frac{\partial^3}{\partial\alpha\partial\beta^2}\right]w-\frac{h^2}{10}\left[a_{55}\left(B_{11}\frac{\partial^2}{\partial\alpha^2}+B_{66}\frac{\partial^2}{\partial\beta^2}\right)\varphi+\right. \\
\left.+a_{44}(B_{12}+B_{66})\frac{\partial^2\psi}{\partial\alpha\partial\beta}\right]+\varphi=0
\end{gathered}
$$

$$\left[(B_{12}+2B_{66})\frac{\partial^4}{\partial\alpha^2\partial\beta^2}+B_{22}\frac{\partial^4}{\partial\beta^4}\right]w-$$

$$-\frac{h^2}{10}\left[a_{55}(B_{12}+B_{66})\frac{\partial^2\varphi}{\partial\alpha\partial\beta}+a_{44}\left(B_{66}\frac{\partial^2}{\partial\alpha^2}+B_{22}\frac{\partial^2}{\partial\beta^2}\right)\psi\right]+\psi=0 \qquad (2.2)$$

cont.

Let us investigate the stability of a rectangular plate supported along the edges, compressed in both principal directions (Fig. 1). Assuming $P_1=P, P_2=\lambda P$, we obtain

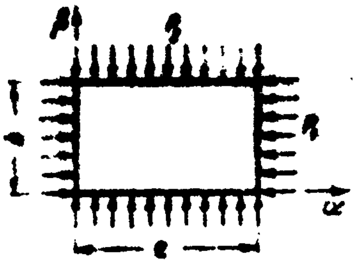

Fig. 1

$$T_1^\circ=-P,\quad T_2^\circ=-\lambda P,\quad S^\circ=0 \qquad (2.3)$$

We seek the solution of system (2.2) in the form of where

$$w=w_0\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}$$
$$\varphi=\varphi_0\cos\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b} \qquad (2.4)$$
$$\psi=\psi_0\sin\frac{m\pi\alpha}{a}\cos\frac{n\pi\beta}{b}$$

w_0, φ_0, ψ_0--unknown constants. Using (2.4) we satisfy the requirements of hinged support around the whole outline of the plate.

Substituting (2.4) into (2.2) we get

$$P\pi^2\left(\frac{m^2}{a^2}+\lambda\frac{n^2}{b^2}\right)w_0-\frac{h^2}{12}\frac{\pi m}{a}\varphi_0-\frac{h^2}{12}\frac{\pi n}{b}\psi_0=0$$

$$\frac{\pi^3 m}{a}\left[B_{11}\frac{m^2}{a^2}+(B_{12}+2B_{66})\frac{n^2}{b^2}\right]w_0-\left[1+a_{55}\frac{\pi^2h^2}{10}\left(B_{11}\frac{m^2}{a^2}+B_{66}\frac{n^2}{b^2}\right)\right]\varphi_0-$$
$$-a_{44}\frac{\pi^2h^2}{10}(B_{12}+B_{66})\frac{mn}{ab}\psi_0=0 \qquad (2.5)$$

$$\frac{\pi^3 n}{b}\left[(B_{12}+2B_{66})\frac{m^2}{a^2}+B_{22}\frac{n^2}{b^2}\right]w_0-a_{55}\frac{\pi^2h^2}{10}(B_{12}+B_{66})\frac{mn}{ab}\varphi_0-$$
$$-\left[1+a_{44}\frac{\pi^2h^2}{10}\left(B_{66}\frac{m^2}{a^2}+B_{22}\frac{n^2}{b^2}\right)\right]\psi_0=0$$

This system has solutions different from zero only in cases when the determinant composed of the coefficients of the system is zero. Equating the determinant of the

system (2.5) to zero we can find the following expression for the critical value of force

$$P_{mn}^{*} = P_{mn}^{\circ}(1+d) \tag{2.6}$$

where

$$P_{mn}^{\circ} = \frac{\pi^2}{m^2/a^2+\lambda n^2/b^2}\left[D_{11}\frac{m^4}{a^4}+2(D_{12}+2D_{66})\frac{m^2n^2}{a^2b^2}+D_{22}\frac{n^4}{b^4}\right] \tag{2.7}$$

is the value of critical force of an analogus problem, assuming validity of Kirchoff's hypothesis /4/

$$d=\frac{A-B}{1+B}, \qquad D_{ik}=\frac{h^3B_{ik}}{12} \tag{2.8}$$

$$A=\frac{\pi^2h^2}{10}\left(a_{44}\frac{m^2}{a^2}+a_{55}\frac{n^2}{b^2}\right)\frac{\left(B_{11}\frac{m^2}{a^2}+B_{66}\frac{n^2}{b^2}\right)\left(B_{66}\frac{m^2}{a^2}+B_{22}\frac{n^2}{b^2}\right)-(B_{12}+B_{66})^2\frac{m^2n^2}{a^2b^2}}{B_{11}\frac{m^4}{a^4}+2(B_{12}+2B_{66})\frac{m^2n^2}{a^2b^2}+B_{22}\frac{n^4}{b^4}}$$

$$B=\frac{\pi^2h^2}{10}\left[(a_{55}B_{11}+a_{44}B_{66})\frac{m^2}{a^2}+(a_{55}B_{66}+a_{44}B_{22})\frac{n^2}{b^2}\right]+$$
$$+a_{44}a_{55}\frac{\pi^4h^4}{100}\left[\left(B_{11}\frac{m^2}{a^2}+B_{66}\frac{n^2}{b^2}\right)\left(B_{66}\frac{m^2}{a^2}+B_{22}\frac{n^2}{b^2}\right)-(B_{12}+B_{66})^2\frac{m^2n^2}{a^2b^2}\right]$$

The first term of formula (2.6) represents the value of the critical force of the investigated problem, determined by the classic theory of plates, but the second one--its correction.

It should be pointed out that the second term of the expression (2.6), which is conventionally called its correction, in some cases may be larger than the basic term determined by the classic theory.

When the plate is compressed in the main direction only, the formula (2.6) remains unchanged, but the expression

$$P_{mn}^{\circ} = \frac{\pi^2a^2}{m^2}\left[D_{11}\frac{m^4}{a^4}+2(D_{12}+2D_{66})\frac{m^2n^2}{a^2b^2}+D_{22}\frac{n^4}{b^4}\right] \tag{2.9}$$

must be used for P_{mn}° instead of (2.7).

In the special case of a rectangular plate supported all around the edges, compressed in direction α, and made of transversely isotropic material (it being assumed that the isotropy planes at each point are parallel to the median plane of the plate /4/ we get for the critical force

$$P_{mn}^{*} = \frac{\pi^2D}{b^2}\,\frac{(m/c+cn^2/m)^2}{1+k(m^2/c^2+n^2)} \tag{2.10}$$

where

$$D = \frac{Eh^3}{12(1-\mu^2)}, \qquad c = \frac{a}{b}, \qquad k = \frac{\pi^2 h^2}{10b^2}\frac{E}{G'(1-\mu^2)} \tag{2.11}$$

E, μ --elasticity modulus and Poissons ratio in the isotropy plane; G'--shear modulus, characterizing the distortion of angles between the directions in the plane of isotropy and perpendicular to it.

In examining (2.10), it is easy to prove that the value of the critical force is minimal when only one half-wave is generated in the direction perpendicular to the direction of compression, that is if $n = 1$. We get

$$P_m^* = \frac{\pi^2 D}{b^2}\frac{(m/c + c/m)^2}{1 + k(1 + m^2/c^2)} \tag{2.12}$$

From (2.12) it may be seen that, as in the analogous problem of the classical plate stability theory /4,5/, the minimum value of critical force P_m^* is independent of m and equals

$$P_{min} = \frac{\pi^2 D}{b^2}\frac{4}{(1+k)^2} \tag{2.13}$$

The values of the plate side ratio $c = a/b$ for different m, when the critical force reaches its minimum (2.13) can be determined from the formula

$$c = m\sqrt{(1-k)/(1+k)} \tag{2.14}$$

As may be seen from (2.12) to (2.14) the determined values substantially depend on the coefficient k, or on the ratio of elasticity constants E/G', Poissons ratio and the ratio h/b, that is on some relative normalized plate thickness, which depends not only on the geometrical parameters, but also on the physical properties of the plate material.

In Table 1 values of coefficient k and coordinates of some characteristic points of curves $\Phi = \Phi(c)$ for $\mu = 0.25$ are given. In the upper part of the Table h/b = 0.1, but in the lower part h/b = 0.2; c_n and Φ_n are points of intersection of stability curves.

Table 1

		Points min		m = 1		m = 2		m = 3	
[illegible]	k	c / m	Φ_{min}	c_n	Φ_n	c_n	Φ_n	c_n	Φ_n
0	0	1.0	4.00	1.414	4.50	2.449	4.167	3.464	4.083
2.5	0.02632	0.974	3.798	1.373	4.245	2.383	3.947	3.372	3.873
5	0.05264	0.949	3.610	1.332	4.008	2.319	3.745	3.283	3.677
10	0.1053	0.900	3.274	1.256	3.593	2.194	3.383	3.110	3.329
0	0	1.0	4.0	1.415	4.50	2.449	4.167	3.464	4.083
2.5	0.1053	0.900	3.274	1.256	3.593	2.194	3.383	3.110	3.329
5	0.2106	0.808	2.730	1.113	2.930	1.961	2.799	2.785	2.765
10	0.4211	0.638	1.981	0.857	2.054	1.536	2.007	2.192	1.994

Based on data of Table 1 stability curves for some values of parameter k are plotted in Fig. 2. Values from curves and Table for $k = 0$ correspond to the solution of the present problem of the classical method. From the examined numerical examples we see that with an increase of parameter k the critical force decreases, as compared with its value obtained by the classical theory of plates ($k = 0$). In addition, the limits of every type of stability narrow down with increase of k .

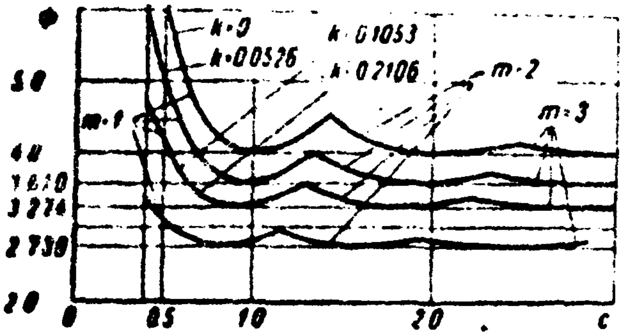

Fig. 2

For the case of an orthotropic plate the calculation of the critical force per formula (2.6) is very cumbersome. Table 2 shows some results of variations calculated with a computer 1). Values of the expression $\Phi(c) = P^{*} b^2/\pi^2 D_{11}$ for different ratios b/a and

$$\frac{E_1}{E_2} = \frac{\mu_1}{\mu_2} = \frac{E_1}{B_{66}} = 5, \quad k_1 = a_{55}E_1 = a_{44}E_2, \quad \frac{h}{b} = \frac{1}{10}, \quad \mu_1 = 0.3$$

are given in the Table. Here E_1, μ_1 and E_2, μ_2 --elasticity moduli and Poissons ratios along directions α and β , respectively.

Table 2

c	$k_1=2$	$=5$	$=10$	[illegible]
0.25	12.6193456	9.1525088	6.2856576	16,9181000
0.45	5.2574384	4.5388992	3.7037616	5,8843716
0.65	3.1438624	9.8728144	2.5156152	3.3109840
0.85	2.3224992	2.1740704	1.9668984	2.4341830
1.05	1.9577536	1.8552720	1.7077264	2.0331295
1.25	1.7985344	1.7164560	1.5960576	1.8581000
1.45	1.7495952	1.6771424	1.5695656	1.8017243
1.65	1.7685984	1.7003600	1.5981904	1.8174095
1.85	1.8344560	1.7673152	1.6661854	1.8822841
2.05	1.9357216	1.8676672	1.7647040	1.0940556
2.25	2.0657552	1.9953648	1.8885064	2.1156309
2.45	2.2204896	2.1466720	2.0343192	2.2726873
	$c=1.4820312$	$c=1.4629392$	$c=1.4335692$	$c=1.5953488$
	$\Phi_{min}=1.7487856$	$\Phi_{min}=1.6770192$	$\Phi_{min}=1.5693776$	$\Phi_{min}=1.8000$[illegible]

Calculations were made only for m- 1.[1] According to the methods based on the case for $m=1$ the cases of $m=2,3..$ can be determined. In the last column of Table 2 values of $\Phi(c)$, calculated according to the classical theory of plates, are given. The two lowest lines give values of c and the corresponding Φ_{min}.

1) The calculations were made on an electronic computer type M-3 at the computing center of the Armenian SSSR.

3. In order to obtain equations of free oscillations of non loaded plates it is necessary to assume for Z /4,6/ in (1.2)

$$Z=-\frac{\gamma_0 h}{g}\frac{\partial^2 w}{\partial t^2}$$

The final equation of free oscillations of the orthotropic plate can be written in the form

$$\frac{\partial\varphi}{\partial\alpha}+\frac{\partial\psi}{\partial\beta}-\frac{12}{h^3}\frac{\gamma_0 h}{g}\frac{\partial^2 w}{\partial t^2}=0 \tag{3.1}$$

$$\left[B_{11}\frac{\partial^3}{\partial\alpha^3}+(B_{12}+2B_{66})\frac{\partial^3}{\partial\alpha\,\partial\beta^2}\right]w-\frac{h^2}{10}\left[a_{55}\left(B_{11}\frac{\partial^2}{\partial\alpha^2}+B_{66}\frac{\partial^2}{\partial\beta^2}\right)\varphi+\right.$$
$$\left.+a_{44}(B_{12}+B_{66})\frac{\partial^2\psi}{\partial\alpha\,\partial\beta}\right]+\varphi=0$$

$$\left[(B_{12}+2B_{66})\frac{\partial^3}{\partial\alpha^2\partial\beta}+B_{22}\frac{\partial^3}{\partial\beta^3}\right]w-\frac{h^2}{10}\left[a_{55}(B_{12}+B_{66})\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta}+\right.$$
$$\left.+a_{44}\left(B_{66}\frac{\partial^2}{\partial\alpha^2}+B_{22}\frac{\partial^2}{\partial\beta^2}\right)\psi\right]+\psi=0$$

where γ_0 -specific gravity of the plate material, g - gravity acceleration. The solution of system (3.1) we seek in the form

$$w = w_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \cos\omega t$$

$$\varphi = \varphi_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \cos\omega t, \qquad \psi = \psi_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \cos\omega t \qquad (3.2)$$

satisfying conditions of free support along the whole contour of the plate.

Substituting (3.2) into (3.1) we obtain a system of homogenous equations in w_0, φ_0, ψ_0. Equating its determinant to zero we get the frequencies of free oscillations from

$$\omega_{mn} = \omega^*_{mn}\sqrt{1+d} \qquad (3.3)$$

where d corresponds to (2.8), and

$$\omega^*_{mn} = \pi^2\sqrt{\frac{g}{\gamma_0 h}}\left[D_{11}\frac{m^4}{a^4} + 2(D_{12}+2D_{66})\frac{m^2n^2}{a^2b^2} + D_{22}\frac{n^4}{b^4}\right]^{1/2} \qquad (3.4)$$

represents, properly speaking, its frequencies, determined according to the classical theory of plates.

If the plate is made of transversally-isotropic material (see section 2) /4/ we get for the frequencies of free oscillations

$$\omega_{mn} = \omega^*_{mn}\frac{1}{\sqrt{1+k(m^2/c^2+n^2)}}, \qquad \omega^*_{mn} = \frac{\pi^2}{b^2}\sqrt{\frac{gD}{\gamma_0 h}}(m^2/c^2+n^2) \qquad (3.5)$$

Table 3 gives values of ratio $\omega_{mn}/\omega^*_{mn}$ corresponding to some oscillation tones ($m, n = 1, 2$) calculated according to formula (3.5) for a square plate ($c = 1$) with different values of parameter k .

Table 3

k	$\omega_{11}/\omega^*_{11}$	$\omega_{12}/\omega^*_{12}$	$\omega_{22}/\omega^*_{22}$	k	$\omega_{11}/\omega^*_{11}$	$\omega_{12}/\omega^*_{12}$	$\omega_{22}/\omega^*_{22}$
0	1	1	1	0.15	0.8771	0.7559	0.6742
0.02	0.9806	0.9535	0.9285	0.20	0.8452	0.7071	0.6202
0.03	0.9713	0.9325	0.8980	0.25	0.8165	0.6667	0.5774
0.05	0.9535	0.8944	0.8452	0.30	0.7906	0.6325	0.5423
0.07	0.9366	0.8607	0.8006	0.35	0.7670	0.6030	0.5130
0.10	0.9121	0.81165	0.7454	0.40	0.7454	0.5774	0.4880

This table shows that with increase of parameter k the frequencies of its oscillations ω_{mn} vary more greatly from the corresponding frequencies ω_{mn}^{0} obtained by the classical theory of plates. This difference, less pronounced at the lowest oscillation frequency (ω_{11}), becomes more substantial at the highest oscillation modes (ω_{12} , etc.)

4. The equation of dynamic stability of an orthotropic plate is found when substituting in (1.2) for Z the expression /7/

$$Z = T_1^{\circ} \frac{\partial^2 w}{\partial \alpha^2} + T_2^{\circ} \frac{\partial^2 w}{\partial \beta^2} + 2S^{\circ} \frac{\partial^2 w}{\partial \alpha \, \partial \beta} - \frac{\gamma_0 h}{g} \frac{\partial^2 w}{\partial t^2} \tag{4.1}$$

Substituting (4.1) into (1.2) we can easily obtain the final equations of dynamic stability of orthotropic plates.

If the plate is compressed in the main direction α only (Fig. 1) then

$$T_1^{\circ} = -P, \qquad T_2^{\circ} = S^{\circ} = 0 \tag{4.2}$$

Let us assume that the external force P varies periodically in time

$$P = P_0 \cos \theta t \tag{4.3}$$

where P_0 - amplitude and θ - frequency of external force. Considering (4.1) to (4.3) the system (1.2) is written as

$$\frac{h^3}{12}\left(\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta}\right) - P_0 \cos \theta t \frac{\partial^2 w}{\partial \alpha^2} - \frac{\gamma_0 h}{g} \frac{\partial^2 w}{\partial t^2} = 0$$

$$\left[B_{11} \frac{\partial^3}{\partial \alpha^3} + (B_{12} + 2B_{66}) \frac{\partial^3}{\partial \alpha \, \partial \beta^2}\right] w - \frac{h^2}{10}\left[a_{55}\left(B_{11} \frac{\partial^2}{\partial \alpha^2} + B_{66} \frac{\partial^2}{\partial \beta^2}\right)\varphi + a_{44}(B_{12} + B_{66}) \frac{\partial^2 \psi}{\partial \alpha \, \partial \beta}\right] + \varphi = 0 \tag{4.4}$$

$$\left[(B_{12} + 2B_{66}) \frac{\partial^3}{\partial \alpha^2 \partial \beta} + B_{22} \frac{\partial^3}{\partial \beta^3}\right] w - \frac{h^2}{10}\left[a_{55}(B_{12} + B_{66}) \frac{\partial^2 \varphi}{\partial \alpha \, \partial \beta} + a_{44}\left(B_{66} \frac{\partial^2}{\partial \alpha^2} + B_{22} \frac{\partial^2}{\partial \beta^2}\right)\psi\right] + \psi = 0$$

We seek solution of system (2.3) in the form of

$$w = w(t) \sin \frac{m\pi \alpha}{a} \sin \frac{n\pi \beta}{b}$$

$$\varphi = \varphi(t) \cos \frac{m\pi \alpha}{a} \sin \frac{n\pi \beta}{b}, \qquad \psi = \psi(t) \sin \frac{m\pi \alpha}{a} \cos \frac{n\pi \beta}{b} \tag{4.5}$$

where $w(t)$, $\varphi(t)$, and $\psi(t)$ --values of cunctions w, φ and ψ in the center of the plate. This will satisfy the conditions of hinged support all around the outline of the plate.

Substituting (4.5) into (4.4) we obtain

$$\frac{\pi^2 m}{a^2} P_0 \cos\theta t\, w(t) - \frac{h^3}{12}\frac{\pi m}{a}\varphi(t) - \frac{h^3}{12}\frac{\pi n}{b}\psi(t) = \frac{\gamma h}{g}\frac{d^2 w(t)}{dt^2}$$

$$\frac{\pi^3 m}{a}\left[B_{11}\frac{m^2}{a^2} + (B_{12}+2B_{66})\frac{n^2}{b^2}\right]w(t) - \left[1+\frac{\pi^2 h^2}{10}a_{55}\left(B_{11}\frac{m^2}{a^2}+B_{66}\frac{n^2}{b^2}\right)\right]\varphi(t) -$$

$$-\frac{\pi^2 h^2}{10}a_{44}(B_{12}+B_{66})\frac{mn}{ab}\psi(t) = 0 \qquad (4.6)$$

$$\frac{\pi^3 n}{b}\left[(B_{12}+2B_{66})\frac{m^2}{a^2} + B_{66}\frac{n^2}{b^2}\right]w(t) - \frac{\pi^2 h^2}{10}a_{55}(B_{12}+B_{66})\frac{mn}{ab}\varphi(t) -$$

$$-\left[1+\frac{\pi^2 h^2}{10}a_{44}\left(B_{66}\frac{m^2}{a^2}+B_{22}\frac{n^2}{b^2}\right)\right]\psi(t) = 0$$

The last two equations of system (4.6), as may be seen, do not contain derivatives with respect to time.

Excluding $\varphi(t)$ and $\psi(t)$ from system (4.6) with respect to $w(t)$ we obtain the differential equation

$$\frac{d^2 w(t)}{dt^2} + \omega_{mn}^2\left(1 - \frac{P_0}{P^*_{mn}}\cos\theta t\right)w(t) = 0 \qquad (4.7)$$

where ω_{mn}, P^*_{mn} -- frequency of free oscillations of the nonloaded plate (3.3) and critical value of the compressing force (2.9) (if the compressing force acts in one direction α only), respectively.

Equation (4.7) is the well-known equation of Matthieu. For certain coefficient relationships it has increasingly unlimited solutions. These solutions fill out complete areas of the parameter plane, to which correspond regions of dynamic instability.

Let us rewrite equation (4.7) as

$$\frac{d^2 w(t)}{dt^2} + \omega_{mn}^2(1 - 2\lambda_{mn}\cos\theta t)\, w(t) = 0 \qquad \left(\lambda_{mn} = \frac{P_0}{2P^*_{mn}}\right) \qquad (4.8)$$

As is known /7/, the boundaries of instability regions can be approximately determined from the following formulas:

$$\theta^{*} = 2\omega_{mn}\sqrt{1 \pm \lambda_{mn}} \qquad (4.9)$$

for the first or main instability region, or

$$\theta^{*} = \omega_{mn}\sqrt{1 + \frac{1}{3}\lambda_{mn}^{2}}\,, \qquad \theta^{*} = \omega_{mn}\sqrt{1 - 2\lambda_{mn}^{2}} \qquad (4.10)$$

for the second instability region, or

$$\theta^{*} = \frac{2}{3}\omega_{mn}\sqrt{1 - \frac{9\lambda_{mn}^{2}}{8 \pm 9\lambda_{mn}}} \qquad (4.11)$$

for the third instability region.

Here θ^{0} are critical frequencies of external load, that is external load frequencies, corresponding to the boundaries of instability regions.

As may be seen from the above formulas, the critical forces, the frequencies and instability region boundaries will be substantially different, depending on the number of half-waves (m,n) in the directions α and β.

There will be a further examination of the case when, in both directions α and β only one half-wave will be generated, that is in all the above formulas we shall take

$$m = n = 1 \qquad (4.12)$$

and for simplicity the indices " 11" shall be omitted.

Let us consider the case when the rectangular plate, supported around the outline, is made of transversally - isotropic material. As before, we assume the plane of isotropy at every point to be parallel with the median plane of the plate.

Table 4

λ^{*}	I region		II region		III region	
For $k=0$ ($\lambda^{*}_{11m}=1/2$)						
0	1.00	1.00	0.50	0.50	0.3333	0.3333
0.1	0.9487	1.0488	0.4950	0.5006	0.3312	0.3316
0.2	0.8944	1.0954	0.4796	0.5033	0.3235	0.3272
0.3	0.8367	1.1402	0.4528	0.5074	0.3068	0.3205
0.4	0.7746	1.1832	0.4123	0.5132	0.2734	0.3120
λ^{*}_{11m}	0.7071	1.2248	0.3536	0.5204	0.1992	0.3018

(Table 4 Cont.)

λ°	I region		II region		III region	
			For $k = 0.05$ ($\lambda^\circ_{lim} = 5/11$)			
0	0.9535	0.9535	0.4767	0.4767	0.3178	0.3178
0.1	0.8995	1.0045	0.4709	0.4777	0.3153	0.3169
0.2	0.8422	1.0531	0.4531	0.4806	0.3061	0.3110
0.3	0.7804	1.0996	0.4216	0.4853	0.2852	0.3033
0.4	0.7135	1.1442	0.3732	0.4919	0.2397	0.2938
λ°_{lim}	0.6742	1.1677	0.3371	0.4962	0.1899	0.2878
			For $k = 0.10$ ($\lambda^\circ_{lim} = 5/12$)			
0	[illegible]	0.9129	0.4564	0.4564	0.3043	0.3043
0.1	0.8583	0.9661	0.4498	0.4575	0.3014	0.3021
0.2	0.7958	1.0165	0.4293	0.4608	0.2905	0.2964
0.3	0.7303	1.0645	0.3929	0.4667	0.2644	0.2881
0,4	0.6583	1.1105	0.3352	0.4736	0.2010	0.2755
λ°_{lim}	0.6157	1.1180	0.3078	0.4751	0.1819	0.2755
			For $k = 0.20$ ($\lambda^\circ_{lim} = 5/14$)			
0	0.8452	0.8452	0.4226	0.4226	0,2817	0.2817
0,1	0.7818	0.9024	0.4142	0.4240	0.2730	0.2790
1.2	0,7171	0.9562	0.3889	0.4281	0.2630	0.2721
0.3	0.6436	1.0071	0.3400	0.4348	0.2225	0.2626
λ°_{lim}	0.5976	1.0551	0.2988	0.4398	0.1684	0.2551
			For $k = 0.30$ ($\lambda^\circ_{lim} = 5/16$)			
0	0.7906	0.7906	0.3953	0.3953	0.2635	0.2635
0.1	0.7246	0.8515	0.3850	0.3970	0.2589	0.2603
0.2	0.6519	0.9083	0.3525	0.4020	0.2720	0.2521
0.3	0.5701	0.9618	0.2903	0.4102	0.1741	0.2403
λ°_{lim}	0.5590	0.9682	0.2795	0.4114	0.1575	0.2386

The regions of dynsmic instability will be determined per formulas (4.9) to (4.11), the indices "mn" being omitted.

Let us examine a square plate (b = a). In order to compare the results obtained for the instability regions with the respective results obtained by the classical theory of plates, -- formulas (4.9) to (4.11) -- taking (4.12) into consideration -- are presented in the following form

$$\frac{\theta^\circ}{2\omega^\circ} = \sqrt{\frac{1}{1+2k} \pm \lambda^\circ} \qquad (4.13)$$

for the first (main) instability region, or

$$\frac{\theta^\circ}{2\omega^\circ} = \frac{1}{2}\sqrt{\frac{1}{1+2k}+\frac{1+2k}{3}(\lambda^\circ)^2}, \quad \frac{\theta^\circ}{2\omega^\circ} = \frac{1}{2}\sqrt{\frac{1}{1+2k}-2(1+2k)(\lambda^\circ)^2} \qquad (4.14)$$

for the second instability region, or

$$\frac{\theta^\circ}{2\omega^\circ} = \frac{1}{3}\sqrt{\frac{1}{1+2k}-\frac{9(1+2k)(\lambda^\circ)^2}{8\pm 9(1+2k)\lambda^\circ}} \qquad \left(\lambda^\circ = \frac{P_0}{2P^*}\right) \qquad (4.15)$$

for the third instability region.

From (4.8) we draw the conclusion that $\lambda \leqslant 1/2$; and therefore for the limit value of λ° we have

$$\lambda^\circ_{\text{limit}} = \frac{1}{2(1+2k)} \qquad (4.16)$$

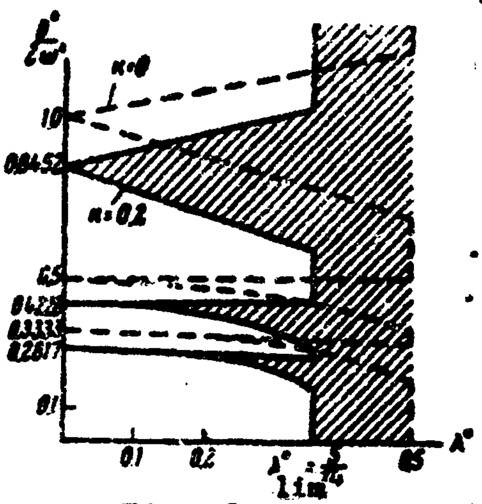

Fig. 3

Table 4 gives values of $\theta^\circ / 2w^\circ$ dependent on λ°, calculated according to formulas (4.13) to (4.15) for different values of parameter k . As illustration Fig. 3 shows regions of instability for k=0 and k=0.2.

In examining Fig. 3 and the above Tables we see that with increase of the coefficient k the instability regions calculated according to (4.13) to (4.15) are different from those calculated according to the classical theory of plates (k=0), this difference being a decrease of $\theta^\circ / 2w^\circ$ and of the interval of parameter .

Analogous calculations for orthotropic square plates (b=a) were made on an electronic computer type M-3 at the computing center of the Acad. Sci. Armenian SSR.

Three variations of these calculations are given in Table 5, showing $\theta^\circ / 2w^\circ$ for different values of parameter k .

Table 5

λ^*	I region		II region		III region	
	For $k_1=10$. $k_2=2$ ($\lambda^*_{lim}=0,4840484$)					
0	0.9839190	0.9839190	0.4919595	0.4919595	0.3279730	0.3279730
0.10	0.9317170	1.0334876	0.4866820	0.4928336	0.3257381	0.3262047
0.20	0.8764112	1.0807852	0.4704945	0.4954466	0.3175495	0.3215206
0.30	0.8173718	1.1260980	0.4422004	0.4997713	0.2995431	0.3145625
0.40	0.7537218	1.1696566	0.3952340	0.5057637	0.2626090	0.3057170
λ^*_{lim}	0.6957358	1.2050498	0.3478679	0.5120477	0.1960013	0.2969922
	$k_1=k_2=5$ ($\lambda^*_{lim}=0.4543373$)					
0	0.9532442	0.9532442	0.4766221	0.4766221	0.3177480	0.3177480
0.10	0.89[illegible]32	1.0043278	0.4708143	0.4775832	0.3152678	0.3158160
0.20	0.8418280	1.0529360	0.4529443	0.4804549	0.3060235	0.3107304
0.30	0.7801758	1.0993972	0.4214806	0.4852034	0,2850747	0.3032098
0.40	0.7132142	1.1439730	0.3729991	0.4917743	0.2394993	0.2936726
λ^*_{lim}	0.6740454	1.1674810	0.3370227	0.4960840	0.1898907	0.2877331
	$k_1=2$, $k_2=10$ ($\lambda^*_{lim}=0.3949343$)					
0	0.8887454	0.8887454	0.4443727	0.4443727	0.2962485	0.2962485
0.10	0.8305832	0.9433284	0.4371921	0.4455582	0.2931173	0.2939012
0.20	0.7680290	0.9949214	0.4149053	0.4490969	0.2809120	0.2878131
0.30	0.6999060	1.0439676	0.3748275	0.4549311	0.2507869	0.2788981
λ^*_{lim}	0.6284378	1.0884864	0.3142189	0.4625178	0.1770423	0.2682644

Notations used in these Tables are

$$k_1 = \frac{E_1}{E_2} = \frac{\mu_1}{\mu_2} = \frac{E_1}{G_{12}}, \qquad k_2 = a_{55}E_1 = a_{44}E_2$$

where E_1, μ_1 and E_2, μ_2 --elasticity moduli and Poissons ratios in directions α and β, respectively; $G_{12} = B_{66}$.

Calculations were made for $\mu = 0.3$, $h/a=0.1$.

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Bibliography

1. Ambartsumyan S. A., On the theory of bending of anisotropic plates, Izv. AN SSSR, OTN, No. 5, 1958.
2. Mushtari Kh. M., Theory of bending of medium thickness plates, Izv. AN SSSR, OTN, No. 2, 1959.
3. Lekhnitskiy S. G., On the theory of thick anisotropic plates, Izv. AN SSSR, OTN, No. 2, 1959.
4. Lekhnitskiy S. G., Anisotropic plates, Gostekhizdat, 1957.
5. Timoshenko S. P., Stability of resilient systems, Gostekhizdat, 1946.
6. Filippov A. P., Oscillations of resilient systems, Izd. AN Ukrainian SSR, 1956.
7. Bolotin V. V., Dynamic stability of resilient systems, Gostekhizdat, 1956.

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